

# Constellation Precoded Beamforming

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## Abstract

We present and analyze the performance of constellation precoded beamforming. This multi-input multi-output transmission technique is based on the singular value decomposition of a channel matrix. In this work, the beamformer is precoded to improve its diversity performance. It was shown previously that while single beamforming achieves full diversity without channel coding, multiple beamforming results in diversity loss. In this paper, we show that a properly designed constellation precoder makes uncoded multiple beamforming achieve full diversity order. We also show that partially precoded multiple beamforming gets better diversity order than multiple beamforming without constellation precoder if the subchannels to be precoded are properly chosen. We propose several criteria to design the constellation precoder. Simulation results match the analysis, and show that precoded multiple beamforming actually outperforms single beamforming without precoding at the same system data rate while achieving full diversity order.

## I. INTRODUCTION

When the perfect channel state information is available at the transmitter, beamforming is employed to achieve spatial multiplexing and thereby increase the data rate, or to enhance the performance of a multi-input multi-output (MIMO) system [1]. The beamforming vectors are designed in [2], [3] for various design criteria, and can be obtained by singular value decomposition (SVD), leading to a channel-diagonalizing structure optimum in minimizing the average bit error rate (BER) [3]. Uncoded single beamforming, which carries only one symbol at a time, was shown to achieve full diversity order of  $MN$  where  $M$  is the number of receive antennas and  $N$  is the number of transmit antennas [4], [5]. However, uncoded multiple beamforming, which increases the throughput by sending multiple symbols at a time, loses full diversity order over flat fading channels [4], [5].

It is known that an SVD subchannel with larger singular value provides larger diversity gain. During the simultaneous parallel transmission of the symbols on the diagonalized subchannels, the performance is dominated by the subchannel with the smallest singular value. To overcome the degradation of the diversity order of multiple beamforming, bit-interleaved coded multiple beamforming (BICMB) was proposed [6], [7]. This scheme interleaves the codewords through the multiple subchannels with different diversity order, resulting in better diversity order. BICMB can achieve the full diversity order offered by the channel as long as the code rate  $R_c$  and the number of subchannels used  $S$  satisfy the condition  $R_c S \leq 1$  [8], [9]. In this paper, we present an uncoded single and multiple beamforming technique that achieves full diversity order. This technique employs the constellation precoding scheme [10], [11], [12], [13], [14], which is used for space-time or space-frequency block codes to increase the system data rate without losing full diversity order. We show via analysis and simulations that fully precoded multiple beamforming achieves full diversity order even in the absence of any channel coding. For this purpose, we derive an upper bound for the pairwise error probability of the precoded beamforming system. Several criteria to design the precoding matrix are proposed. Simulation results show that fully precoded multiple beamforming with a properly designed precoding matrix outperforms single beamforming at the same system data rate.

The rest of this paper is organized as follows. The description of precoded beamforming is given in Section II. Section III calculates the upper bound of pairwise error probability for different schemes of precoded beamforming. In Section IV, several criteria to design the precoding matrix are proposed. Simulation results supporting the analysis are shown in Section V. Finally, we end the paper with a conclusion in Section VI.

## II. SYSTEM MODEL

The MIMO channel  $\mathbf{H} \in \mathbb{C}^{M \times N}$  is assumed to be quasi-static, Rayleigh, and flat fading, and perfectly known to both the transmitter and the receiver. The beamforming vectors are determined by the SVD of the MIMO channel, i.e.,  $\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^H$  where  $\mathbf{U}$  and  $\mathbf{V}$  are unitary matrices, and  $\mathbf{\Lambda}$  is a diagonal matrix whose  $s^{th}$  diagonal element,  $\lambda_s \in \mathbb{R}$ , is a singular value of  $\mathbf{H}$  in decreasing order. When  $S$  symbols are transmitted at the same time, then the first  $S$  vectors of  $\mathbf{U}$  and  $\mathbf{V}$  are chosen to be used as beamforming matrices at the receiver and the transmitter, respectively. In Fig. 1 which displays the structure of constellation precoded beamforming,  $\tilde{\mathbf{U}}$  and  $\tilde{\mathbf{V}}$  denote the beamforming matrices picked from  $\mathbf{U}$  and  $\mathbf{V}$ . Depending on  $S$  and the number of symbols precoded  $R$ , constellation precoded beamforming can be classified into three types as are described below.

### A. Precoded Single Beamforming

In the Precoded Single Beamforming (PSB) scheme, a precoder  $\boldsymbol{\theta}^T$  generates a symbol  $y$  from an  $R \times 1$  modulated symbol vector  $\mathbf{x}$ ,  $y = \boldsymbol{\theta}^T \mathbf{x}$ . We assume that each of the  $R$  members of  $\mathbf{x}$  belongs to a normalized signal set  $\chi \subset \mathbb{C}$  of size  $|\chi| = 2^m$ , such as  $2^m$ -QAM, where  $m$  is the number of input bits to the Gray encoder. Due to the beamforming vectors employed at both of the ends, the precoded symbol is transmitted on the subchannel with the largest singular value. Hence, the detected symbol  $r$  is written as

$$r = \lambda_1 \boldsymbol{\theta}^T \mathbf{x} + n \quad (1)$$

where  $\lambda_1$  is the largest singular value of the channel matrix  $\mathbf{H}$ , and  $n$  is an additive white Gaussian noise with zero mean and variance  $N_0 = N/SNR$ .  $\mathbf{H}$  is complex Gaussian with zero mean and unit variance, and to make the received signal-to-noise ratio  $SNR$ , the total transmitted power is scaled as  $N$ . The maximum likelihood (ML) decoding of the detected symbol is given by

$$\hat{\mathbf{x}} = \min_{\mathbf{x} \in \chi^R} |r - \lambda_1 \boldsymbol{\theta}^T \mathbf{x}|^2. \quad (2)$$

The system data rate for PSB is  $\eta = m \cdot R$  bits/channel use.

### B. Precoded Multiple Beamforming

In this scheme,  $S > 1$  modulated symbols are simultaneously transmitted on the subchannels with the largest  $S$  singular values. The  $S \times 1$  symbol vector  $\mathbf{x}$  whose elements belong to  $\chi$  are precoded by a square precoding matrix  $\boldsymbol{\Theta}$ . The constellation precoder can be expressed as

$$\boldsymbol{\Theta} = \mathbf{P} \times \begin{bmatrix} \tilde{\boldsymbol{\Theta}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{S-R} \end{bmatrix} \quad (3)$$

where  $\tilde{\boldsymbol{\Theta}}$  is  $R \times R$  constellation precoding matrix that precodes the first  $R$  modulated symbols of the vector  $\mathbf{x}$ , and  $\mathbf{P}$  is an  $S \times S$  permutation matrix to define the mapping of the precoded and non-precoded symbols onto the predefined subchannels. When all of the  $S$  modulated symbols are precoded ( $R = S$ ), we call the resulting system Fully Precoded Multiple Beamforming (FPMB), otherwise, we call it Partially Precoded Multiple Beamforming (PPMB). For FPMB,  $\mathbf{P}$  can be replaced by the identity matrix, and

$\Theta = \tilde{\Theta}$ . The  $S \times 1$  detected symbol vector  $\mathbf{r}$  at the receiver is written as

$$\mathbf{r} = \Lambda_S \Theta \mathbf{x} + \mathbf{n} \quad (4)$$

where  $\Lambda_S$  is a diagonal matrix whose elements are the first  $S$  singular values of  $\Lambda$ , and  $\mathbf{n}$  is an additive white Gaussian noise vector. The ML decoding of the detected symbol is given by

$$\hat{\mathbf{x}} = \min_{\mathbf{x} \in \chi^S} \|\mathbf{r} - \Lambda_S \Theta \mathbf{x}\|^2. \quad (5)$$

The system data rate for precoded multiple beamforming is  $\eta = m \cdot S$  bits/channel use.

### III. DIVERSITY ANALYSIS

In this section, we will calculate the diversity order by analyzing the pairwise error probability (PEP) between the transmitted symbol  $\mathbf{x} = [x_1 \dots x_S]^T$  and the detected symbol  $\hat{\mathbf{x}} = [\hat{x}_1 \dots \hat{x}_S]^T$ .

#### A. Precoded Single Beamforming

For the ML decoding criterion of (2), the instantaneous PEP can be expressed as

$$\begin{aligned} \Pr(\mathbf{x} \rightarrow \hat{\mathbf{x}} \mid \mathbf{H}) &= \Pr(|r - \lambda_1 \boldsymbol{\theta}^T \mathbf{x}|^2 \geq |r - \lambda_1 \boldsymbol{\theta}^T \hat{\mathbf{x}}|^2 \mid \mathbf{H}) \\ &= \Pr(\beta \geq |\lambda_1 \boldsymbol{\theta}^T (\mathbf{x} - \hat{\mathbf{x}})|^2 \mid \mathbf{H}) \end{aligned} \quad (6)$$

where  $\beta = -\lambda_1 [\boldsymbol{\theta}^T (\mathbf{x} - \hat{\mathbf{x}})]^* n - \lambda_1 n^* \boldsymbol{\theta}^T (\mathbf{x} - \hat{\mathbf{x}})$ . Since  $\beta$  is a zero mean Gaussian random variable with variance  $2N_0 \lambda_1^2 |\boldsymbol{\theta}^T (\mathbf{x} - \hat{\mathbf{x}})|^2$ , (6) is rewritten as

$$\Pr(\mathbf{x} \rightarrow \hat{\mathbf{x}} \mid \mathbf{H}) = Q \left( \sqrt{\frac{\lambda_1^2 |\boldsymbol{\theta}^T (\mathbf{x} - \hat{\mathbf{x}})|^2}{2N_0}} \right) \quad (7)$$

where  $Q(\cdot)$  is the well-known  $Q$  function. By using the upper bound on the  $Q$  function  $Q(x) \leq \frac{1}{2} e^{-x^2/2}$ , the average PEP can be expressed as

$$\begin{aligned} \Pr(\mathbf{x} \rightarrow \hat{\mathbf{x}}) &= E [\Pr(\mathbf{x} \rightarrow \hat{\mathbf{x}} \mid \mathbf{H})] \\ &\leq E \left[ \frac{1}{2} \exp \left( -\frac{\lambda_1^2 |\boldsymbol{\theta}^T (\mathbf{x} - \hat{\mathbf{x}})|^2}{4N_0} \right) \right]. \end{aligned} \quad (8)$$

Assume that  $|\boldsymbol{\theta}^T (\mathbf{x} - \hat{\mathbf{x}})|^2 > 0$  for a distinct pair of  $\mathbf{x}$  and  $\hat{\mathbf{x}}$ . Previously, in [4], [8], and [9], we showed the closed form expression of (8). We provide a formal description of the result from [8], [9] below.

*Theorem 1:* Consider the largest  $S \leq \min(M, N)$  eigenvalues  $\mu_s$  of the uncorrelated central  $M \times N$  Wishart matrix that are sorted in decreasing order, and a weight vector  $\boldsymbol{\alpha} = [\alpha_1 \cdots \alpha_S]^T$  with non-negative real elements. In the high signal-to-noise ratio regime, an upper bound for the expression  $E[\exp(-\gamma \sum_{s=1}^S \alpha_s \mu_s)]$  which is used in the diversity analysis of a number of MIMO systems is

$$E \left[ \exp \left( -\gamma \sum_{s=1}^S \alpha_s \mu_s \right) \right] \leq \zeta (\alpha_{\min} \gamma)^{-(M-\delta+1)(N-\delta+1)} \quad (9)$$

where  $\gamma$  is signal-to-noise ratio,  $\zeta$  is a constant,  $\alpha_{\min} = \min\{\alpha_1, \cdots, \alpha_S\}$ , and  $\delta$  is the index to the first non-zero element in the weight vector.

*Proof:* See [8], [9]. □

Applying Theorem 1 to (8) where  $S = 1$ ,  $\delta = 1$ , and  $\alpha_{\min} = |\boldsymbol{\theta}^T(\mathbf{x} - \hat{\mathbf{x}})|^2$ , we get the upper bound for the PEP as

$$\Pr(\mathbf{x} \rightarrow \hat{\mathbf{x}}) \leq \frac{1}{2} \left( \frac{|\boldsymbol{\theta}^T(\mathbf{x} - \hat{\mathbf{x}})|^2}{4N} SNR \right)^{-MN}. \quad (10)$$

Therefore, it is easily found that PSB achieves full diversity order of  $MN$  once it satisfies the condition

$$|\boldsymbol{\theta}^T(\mathbf{x} - \hat{\mathbf{x}})|^2 > 0 \quad (11)$$

for any distinct pair of  $\mathbf{x}$  and  $\hat{\mathbf{x}}$ . The method to design the precoding vector will be described in Section IV.

### B. Fully Precoded Multiple Beamforming

By using the same approach in PSB, we get the upper bound to the instantaneous PEP for precoded multiple beamforming as

$$\begin{aligned} \Pr(\mathbf{x} \rightarrow \hat{\mathbf{x}} \mid \mathbf{H}) &= \Pr(\|\mathbf{r} - \mathbf{\Lambda}_S \boldsymbol{\Theta} \mathbf{x}\|^2 \geq \|\mathbf{r} - \mathbf{\Lambda}_S \boldsymbol{\Theta} \hat{\mathbf{x}}\|^2 \mid \mathbf{H}) \\ &\leq \frac{1}{2} \exp \left( -\frac{\|\mathbf{\Lambda}_S \boldsymbol{\Theta}(\mathbf{x} - \hat{\mathbf{x}})\|^2}{4N_0} \right). \end{aligned} \quad (12)$$

Let's define  $\mathbf{d} = [d_1 \cdots d_S]^T$  as a Euclidean vector that results from the precoded symbols of a distinct pair  $\mathbf{x}$  and  $\hat{\mathbf{x}}$ . Then,  $\mathbf{d} = \boldsymbol{\Theta}(\mathbf{x} - \hat{\mathbf{x}})$ , and the absolute value  $|d_i|$  can be interpreted as a Euclidean distance between the symbols belonging to a new constellation transformed by the  $i^{th}$  row vector of  $\boldsymbol{\Theta}$  from the

original constellation. For FPMB, the average PEP is expressed as

$$\Pr(\mathbf{x} \rightarrow \hat{\mathbf{x}}) \leq E \left[ \frac{1}{2} \exp \left( -\frac{\sum_{s=1}^S \lambda_s^2 |d_s|^2}{4N_0} \right) \right]. \quad (13)$$

Applying Theorem 1 to (13), we get the upper bound to PEP as

$$\Pr(\mathbf{x} \rightarrow \hat{\mathbf{x}}) \leq \zeta \left( \frac{\hat{d}_{min}}{4N} SNR \right)^{-(M-\delta+1)(N-\delta+1)} \quad (14)$$

where  $\zeta$  is a constant,  $\hat{d}_{min} = \min\{|d_1|^2, \dots, |d_S|^2\}$ , and  $\delta$  is an index to the first non-zero element of the (squared) Euclidean distance vector  $[|d_1|^2 \dots |d_S|^2]$ . Therefore, FPMB also achieves full diversity order if  $\delta$  from any distinct pair is equal to 1, which implies that  $|d_1|^2 = |\boldsymbol{\theta}_1^T(\mathbf{x} - \hat{\mathbf{x}})|^2 > 0$  for any distinct pair, where  $\boldsymbol{\theta}_1^T$  is the first row vector of  $\boldsymbol{\Theta}$ . The way to build the precoding matrix will be described in Section IV.

### C. Partially Precoded Multiple Beamforming

The partial precoding scheme divides the modulated symbols into two groups of symbols, i.e., precoded and non-precoded symbols. Through the permutation and the grouping, the numerator of the exponent term in (12) can be represented as described below. For this purpose, let's define  $\mathbf{b}_p = [b_p(1) \dots b_p(R)]$  as a vector whose element  $b_p(k)$  is the subchannel on which the precoded symbols are transmitted, and  $b_p(k) < b_p(l)$  for  $k < l$ . In the same way,  $\mathbf{b}_n = [b_n(1) \dots b_n(S-R)]$  is defined as an increasingly ordered vector whose element  $b_n(k)$  is the subchannel which carries the non-precoded symbols. By reordering the resulting vector  $\boldsymbol{\Lambda}_S \boldsymbol{\Theta}(\mathbf{x} - \hat{\mathbf{x}})$  for a simpler representation of  $\boldsymbol{\Lambda}_S \mathbf{P}$ , the numerator of the exponent term in (12) can be expressed as

$$\begin{aligned} \|\boldsymbol{\Lambda}_S \boldsymbol{\Theta}(\mathbf{x} - \hat{\mathbf{x}})\| &= \left\| \boldsymbol{\Lambda}_S \mathbf{P} \begin{bmatrix} \tilde{\boldsymbol{\Theta}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{S-R} \end{bmatrix} (\mathbf{x} - \hat{\mathbf{x}}) \right\| \\ &= \left\| \begin{bmatrix} \boldsymbol{\Lambda}_p & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_n \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\Theta}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{S-R} \end{bmatrix} \begin{bmatrix} \mathbf{x}_p - \hat{\mathbf{x}}_p \\ \mathbf{x}_n - \hat{\mathbf{x}}_n \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} \boldsymbol{\Lambda}_p \tilde{\boldsymbol{\Theta}}(\mathbf{x}_p - \hat{\mathbf{x}}_p) \\ \boldsymbol{\Lambda}_n(\mathbf{x}_n - \hat{\mathbf{x}}_n) \end{bmatrix} \right\| \end{aligned} \quad (15)$$

where  $\mathbf{\Lambda}_p$  and  $\mathbf{\Lambda}_n$  are the  $R \times R$  and  $(S - R) \times (S - R)$  diagonal matrices whose elements are the ordered singular values corresponding to the subchannels of the vectors  $\mathbf{b}_p$  and  $\mathbf{b}_n$ , respectively, and similarly  $\mathbf{x}_p = [x_1 \cdots x_R]$ ,  $\mathbf{x}_n = [x_{R+1} \cdots x_S]$ ,  $\hat{\mathbf{x}}_p = [\hat{x}_1 \cdots \hat{x}_R]$ ,  $\hat{\mathbf{x}}_n = [\hat{x}_{R+1} \cdots \hat{x}_S]$ . By plugging (15) in (12), we get an upper bound to PEP for PPMB as

$$\Pr(\mathbf{x} \rightarrow \hat{\mathbf{x}}) \leq E \left[ \frac{1}{2} \exp \left( -\frac{\kappa}{4N_0} \right) \right] \quad (16)$$

where

$$\kappa = \sum_{i=1}^R \lambda_{b_p(i)}^2 |\tilde{d}_i|^2 + \sum_{i=1}^{S-R} \lambda_{b_n(i)}^2 |x_{R+i} - \hat{x}_{R+i}|^2 \quad (17)$$

and  $\tilde{d}_i$  is the  $i^{th}$  element of a Euclidean vector  $\tilde{\mathbf{d}} = \tilde{\mathbf{\Theta}}(\mathbf{x}_p - \hat{\mathbf{x}}_p)$ . Let's assume that the constellation precoding matrix  $\tilde{\mathbf{\Theta}}$  meets the condition of FPMB to achieve full diversity order. Since (17) has the closed form expression similar to (14) as described in FPMB,  $\delta$  value needs to be obtained from a composite vector with such kind of elements as  $|\tilde{d}_i|^2$  and  $|x_{R+i} - \hat{x}_{R+i}|^2$ , to observe the diversity behavior of a given pairwise error. In addition, a different pair can lead to different diversity behavior. Therefore, we need to get the maximum  $\delta$  out of all the possible pairwise errors to decide the diversity order of a given PPMB system.

All of the distinct pairs of  $\mathbf{x}$  and  $\hat{\mathbf{x}}$  can be divided into three groups in terms of  $\mathbf{x}_p$ ,  $\hat{\mathbf{x}}_p$ ,  $\mathbf{x}_n$ , and  $\hat{\mathbf{x}}_n$ . The first group includes the pairs that have  $\mathbf{x}_p = \hat{\mathbf{x}}_p$ , and the second group comprises the pairs satisfying  $\mathbf{x}_p \neq \hat{\mathbf{x}}_p$ , but  $\mathbf{x}_n = \hat{\mathbf{x}}_n$ . Finally, the last group is consisted of the pairs that  $\mathbf{x}_p \neq \hat{\mathbf{x}}_p$ , and  $\mathbf{x}_n \neq \hat{\mathbf{x}}_n$ . We will present the method to calculate  $\delta$  for a pair of each group, and to find  $\delta_{max}$  for each group.

Since the vector  $\tilde{\mathbf{d}}$  is a zero vector for the first group, the first summation of  $\kappa$  in (17) is zero, resulting in  $\delta$  being equal to the minimum of  $\mathbf{b}_n$ . By considering all of the possible pairs, we can easily see that  $b_n(1) \leq \delta \leq b_n(S - R)$ . Therefore, the maximum value is  $\delta_1 = b_n(S - R)$  which corresponds to the pair satisfying  $x_i = \hat{x}_i$  for all  $i$  except  $i = b_n(S - R)$ . For any pair in the second group, the term with the first singular value survives in  $\kappa$ , according to the inherited property of the constellation precoding matrix, i.e.,  $|\tilde{d}_1|^2 > 0$ . However, the second summation in  $\kappa$  disappears since  $\mathbf{x}_n = \hat{\mathbf{x}}_n$ . Therefore, the maximum value of this group  $\delta_2 = b_p(1)$ . Now, for the third group, both summations in  $\kappa$  exist. Then,  $\delta$  is chosen to be the smaller value between the minimum of  $\mathbf{b}_n$  and  $b_p(1)$ . In the same manner as was already given in the analysis of the first group, the maximum of the minimum of  $\mathbf{b}_n$  can be found to be  $b_n(S - R)$ . Therefore, the maximum  $\delta$  for this group is  $\delta_3 = \max\{b_p(1), b_n(S - R)\}$ . Finally,  $\delta_{max}$  can be decided

as

$$\begin{aligned}
\delta_{max} &= \max\{\delta_1, \delta_2, \delta_3\} \\
&= \max\{b_n(S - R), b_p(1), \max\{b_p(1), b_n(S - R)\}\} \\
&= \max\{b_p(1), b_n(S - R)\}.
\end{aligned} \tag{18}$$

**Example:** We provide the diversity analysis of the  $4 \times 4$  partially precoded multiple beamforming system with  $S = 4$  and  $R = 2$ . In this example, we assume that the precoded symbols are transmitted on the subchannel 1 and 3 while the non-precoded symbols are transmitted on the subchannel 2 and 4. Then, this configuration gives  $\mathbf{b}_p = [1 \ 3]$ , and  $\mathbf{b}_n = [2 \ 4]$ . By following the result in (18),  $\delta_{max}$  is equal to  $\max\{1, 4\} = 4$ , leading to the diversity order of 1. The pairwise errors, satisfying  $x_1 = \hat{x}_1, x_2 = \hat{x}_2, x_3 = \hat{x}_3$ , but  $x_4 \neq \hat{x}_4$ , inflict loss on the diversity order of this system. Table I summarizes the diversity order analysis for all of the possible combinations of the  $4 \times 4$  partially precoded multiple beamforming system.

#### IV. PRECODER DESIGN

##### A. Precoded Single Beamforming

The optimum precoding vector should maximize the array gain as well as meet the condition for achieving full diversity order. Based on (10), the optimum precoding vector  $\boldsymbol{\theta}_{opt}$  can be determined by solving the following equation:

$$\boldsymbol{\theta}_{opt} = \arg \max_{\boldsymbol{\theta}} \min_{\forall \mathbf{x} \neq \hat{\mathbf{x}}} |\boldsymbol{\theta}^T (\mathbf{x} - \hat{\mathbf{x}})| \tag{19}$$

subject to the power constraint  $E|\boldsymbol{\theta}^T \mathbf{x}|^2 = E \|\mathbf{x}\|^2 = R$ . The problem in (19) coincides with that of [15] which addressed multi-user coding. In [15], a number of users are assumed to transmit simultaneously at the same power, each of whom uses a rotated version of QAM symbols. Thus, the noiseless received symbol belongs to a new constellation containing the sum of each rotated QAM symbols. For a simple construction, the rotation vector is defined as

$$\boldsymbol{\theta} = \begin{bmatrix} 1 & e^{j\phi} & e^{j2\phi} & e^{j4\phi} & \dots & e^{j2^{R-2}\phi} \end{bmatrix}^T \tag{20}$$

where  $j = \sqrt{-1}$ . The optimum rotation angles for  $R \leq 7$  are found by searching  $\phi$  that maximizes the minimum squared Euclidean distance of the new constellation. Hence, we can apply the result of [15] to



precoded single beamforming.

### B. Precoded Multiple Beamforming

To design the precoding matrix, we establish various design criteria. Since we focus on even power distribution on each subchannel, the precoding matrix is restricted to be a unitary matrix which preserves the power.

1) *Maximization of the Minimum Euclidean Distance Among the  $S$  Coordinates,  $\Phi_1$* : This criterion minimizes the upper bound to PEP of (14) by maximizing  $\hat{d}_{min}$  as

$$\Theta_{opt} = \arg \max_{\Theta} \min_{\forall \mathbf{x} \neq \hat{\mathbf{x}}} \hat{d}_{min} \quad (21)$$

subject to the power constraint  $E\|\Theta\mathbf{x}\|^2 = E\|\mathbf{x}\|^2 = R$ . Since the analytical solution of this problem is unavailable, computer search can be used for small  $S$  and small constellation sizes. For this purpose, we employ the parameterization method of complex unitary matrices in [11]. In this method, any  $S \times S$  unitary matrix  $\Sigma$  can be written as

$$\Sigma = \mathbf{D} \prod_{1 \leq k \leq S-1, k+1 \leq l \leq S} \mathbf{G}_{kl}(\psi_{kl}, \rho_{kl}) \quad (22)$$

where  $\mathbf{D}$  is an  $S \times S$  diagonal unitary matrix,  $\psi_{kl} \in [-\pi, \pi]$ ,  $\rho_{kl} \in [-\pi/2, \pi/2]$ , and  $\mathbf{G}_{kl}(\psi_{kl}, \rho_{kl})$  is a complex Givens matrix, which is the  $S \times S$  identity matrix with the  $(k, k)^{th}$ ,  $(l, l)^{th}$ ,  $(k, l)^{th}$ ,  $(l, k)^{th}$  elements substituted by  $\cos \psi_{kl}$ ,  $\cos \psi_{kl}$ ,  $e^{-j\rho_{kl}} \sin \psi_{kl}$ , and  $-e^{j\rho_{kl}} \sin \psi_{kl}$ , respectively. The optimization of the diagonal unitary matrix  $\mathbf{D}$  is not necessary since  $\hat{d}_{min}$  is the squared absolute value of an element which includes the diagonal entry of  $\mathbf{D}$  with the magnitude equal to one. Therefore, we need to optimize only  $S(S-1)$  parameters of  $\{\psi_{kl}, \phi_{kl}\}$ . The optimum precoding matrix found by this method automatically satisfies the full diversity order condition since  $|d_1|^2 \geq \hat{d}_{min} > 0$ .

2) *Maximization of the Minimum Euclidean Distance of the First Coordinate,  $\Phi_2$* : For large  $S$ , the squared Euclidean distance of  $i^{th}$  coordinate  $|d_i|^2$  is smaller than  $\lambda_1^2$ . Thus, the term  $\lambda_1^2 |d_1|^2$  takes the biggest portion of the summation in (13). This fact leads to an idea that maximization of the minimum Euclidean distance of the first coordinate  $|d_1|^2$  can lower the PEP. In this criterion, we solve the following equation as

$$\Theta_{opt} = \arg \max_{\Theta} \min_{\forall \mathbf{x} \neq \hat{\mathbf{x}}} |\boldsymbol{\theta}_1^T (\mathbf{x} - \hat{\mathbf{x}})|^2 \quad (23)$$

subject to the power constraint  $E\|\Theta\mathbf{x}\|^2 = E\|\mathbf{x}\|^2 = R$ . Since it is difficult to solve (23) in a tractable way, and the optimization equation is the same as (19), we propose a method that adopts the result of (19). In this method, we use the optimum precoding vector (19) in  $\Theta_{opt}$  as the first row vector. To build a unitary matrix, we utilize the  $S$ -point inverse fast Fourier transform (IFFT) matrix  $\mathbf{F}_S$ , whose  $(l, m)$  element is given by  $\sqrt{S}\exp(j2\pi(l-1)(m-1)/S)$ . The IFFT matrix provides two properties we can use; the elements of the first column vector are all ones, and the IFFT matrix is a unitary matrix. By constructing a precoding matrix as

$$\Theta_{opt} = \mathbf{F}_S^T \text{diag}(\hat{\boldsymbol{\theta}}_{opt}) \quad (24)$$

where  $\hat{\boldsymbol{\theta}}_{opt}$  is the precoding vector obtained from (19), we can see that the unitary matrix  $\Theta_{opt}$  contains  $\hat{\boldsymbol{\theta}}_{opt}$  as the first row vector. This method guarantees full diversity order since  $|d_1|^2 > 0$ .

3) *Maximization of the Geometric Mean,  $\Phi_3$* : Since the summation in (13) consists of many terms, the previous optimizations which optimize only one term may not necessarily be the best criterion. Maximizing the arithmetic mean or the geometric mean of  $|d_i|^2$  values are attractive candidates which consider all of  $|d_i|^2$  values simultaneously. Between the two, we choose the geometric mean since the arithmetic mean does not necessarily guarantee the full diversity order condition. However, the geometric mean meets the condition on  $|d_1|^2 > 0$  since the geometric mean of the optimum precoding matrix in this sense will be larger than zero. Therefore, the optimization is based on

$$\Theta_{opt} = \arg \max_{\Theta} \min_{\forall \mathbf{x} \neq \hat{\mathbf{x}}} \prod_{i=1}^S |\boldsymbol{\theta}_i^T (\mathbf{x} - \hat{\mathbf{x}})|^{2/S} \quad (25)$$

subject to the same power constraint as the previous criteria, and  $\boldsymbol{\theta}_i^T$  is the  $i^{th}$  row vector of the precoding matrix  $\Theta$ . Authors in [11] introduced an algebraic method to construct the precoding matrix in the space-time diversity system. According to [11], the unitary precoding matrix can be written as

$$\Theta_{opt} = \mathbf{F}_S^T \text{diag}(1, \sigma, \dots, \sigma^{S-1}) \quad (26)$$

where  $\sigma = e^{j2\pi/P}$ , and the method to determine  $P$  is available in [11].

## V. SIMULATION RESULTS

To illustrate the analysis of the diversity order in Section III, we now present simulation results over various channel dimensions. Fig. 2, 3, and 4 show bit error rate (BER) performance of SB, PSB, and

FPMB. The curves with legend FPMB  $\Phi_1$ ,  $\Phi_2$ ,  $\Phi_3$  are generated by the precoding matrices based on each criterion in Section IV. For a fair comparison between the different schemes, the system data rate  $\eta$  is set to 4, 6, 8 bits/channel use. Throughout the figures, PSB and FPMB are shown to achieve full diversity order since the slopes are parallel to that of SB which is known to achieve full diversity order. A comparison between SB and PSB reveals that SB outperforms PSB for any channel dimension. The reason is that the array gain which can be observed by PEP is related to the minimum squared Euclidean distance, and the minimum squared Euclidean distance of the new constellation generated by the precoding vector is smaller than that of the SB constellation. For example, the minimum squared Euclidean distance of  $\eta = 4$  PSB normalized constellation is 0.27, while that of normalized 16-QAM SB is 0.4.

Contrary to the case of  $2 \times 2$ , FPMB outperforms SB for larger channel dimension. In the case of  $3 \times 3$ , FPMB  $\Phi_3$  gives 2 dB gain over SB. A bigger gain of 6 dB is observed by FPMB  $\Phi_3$  for the case of  $4 \times 4$ . Bigger gain for larger dimension can be explained by a comparison of the instantaneous PEPs of SB in [4] and FPMB in (13). It can be stated that a larger number of singular values lead to a bigger array gain. We also find that the maximization of the geometric mean results in better performance than the others for larger channel dimension.

Simulation results to support the diversity analysis of  $4 \times 4$   $S = 4$  PPMB in Table I are provided in Fig. 5. We find that the simulation results follow the diversity orders in Table I. The curves with the same diversity order give different array gain depending on the subchannel selection of precoded symbol transmission. BER at high SNR for  $\mathbf{b}_p = [1\ 4]$ ,  $[2\ 4]$ ,  $[1\ 2\ 4]$  are the same, leading to around 10 dB larger gain than that of  $\mathbf{b}_p = [3\ 4]$ . This can be explained by the fact that PPMB with the smaller number of pairs (causing the worst diversity order) provide larger array gain. Since the subchannel 3 transmitting the non-precoded symbol dominates the performance loss for  $\mathbf{b}_p = [1\ 4]$ ,  $[2\ 4]$ ,  $[1\ 2\ 4]$ , only one pair, satisfying  $x_3 = \hat{x}_3$  and  $x_i \neq \hat{x}_i$  for  $i = 1, 2, 4$ , is related to the worst diversity order. On the other hand, for  $\mathbf{b}_p = [3\ 4]$  where the subchannel 3 also dominates the performance loss, half of the total possible pairs, such that  $x_1 = \hat{x}_1$ , and  $x_2 = \hat{x}_2$ , and  $x_3 \neq \hat{x}_3$ , or  $x_4 \neq \hat{x}_4$ , make the worst diversity order. This fact also applies to the case of diversity order of 9.

## VI. CONCLUSION

In this paper, we introduced a precoded beamforming system that achieves full diversity order. For the analysis, we calculated an upper bound to the pairwise error probability, assuming that the receiver decodes the transmitted symbols based on maximum likelihood decoding. We established several criteria

to design the precoder. We also provided simulation results that support the diversity analysis of precoded beamforming. We showed in particular fully precoded multiple beamforming with a proper precoding matrix outperforms single beamforming at the same system data rate while achieving full diversity order.

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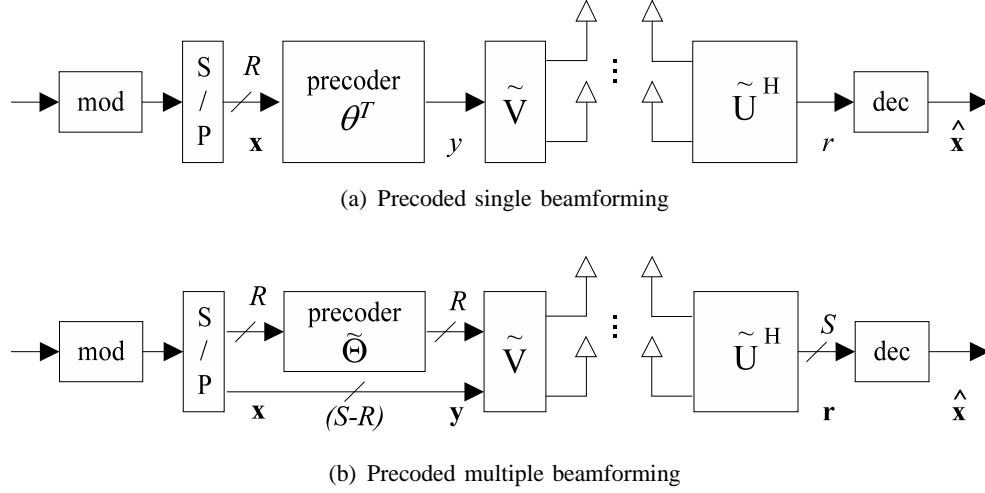


Fig. 1. Structure of Constellation Precoded Beamforming.

TABLE I  
DIVERSITY ORDER OF  $4 \times 4$ ,  $S = 4$  PARTIALLY PRECODED MULTIPLE BEAMFORMING SYSTEM

$R$	$\mathbf{b}_p$	$\mathbf{b}_n$	$b_p(1)$	$b_n(S - R)$	$\delta_{max}$	$O_{div}$
2	[1 2]	[3 4]	1	4	4	1
	[1 3]	[2 4]	1	4	4	1
	[1 4]	[2 3]	1	3	3	4
	[2 3]	[1 4]	2	4	4	1
	[2 4]	[1 3]	2	3	3	4
	[3 4]	[1 2]	3	2	3	4
3	[1 2 3]	[4]	1	4	4	1
	[1 2 4]	[3]	1	3	3	4
	[1 3 4]	[2]	1	2	2	9
	[2 3 4]	[1]	2	1	2	9

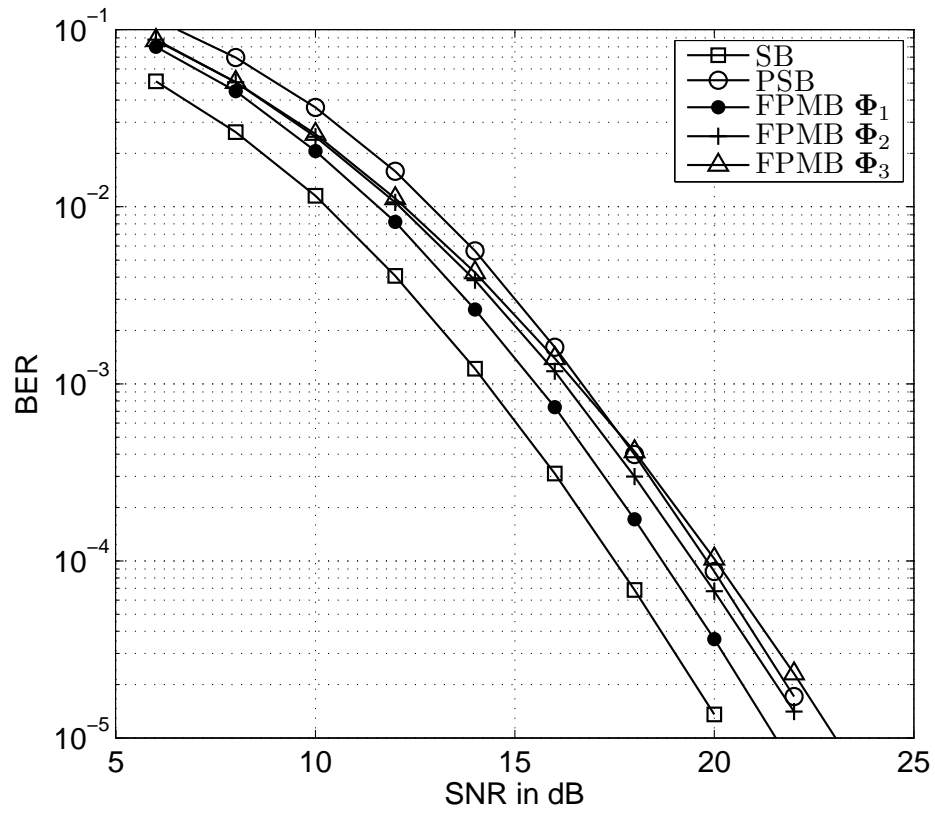


Fig. 2. BER vs. SNR comparison for  $2 \times 2$  16-QAM SB, 4-QAM  $R = 2$  PSB, and 4-QAM  $S = 2$  FPMB.

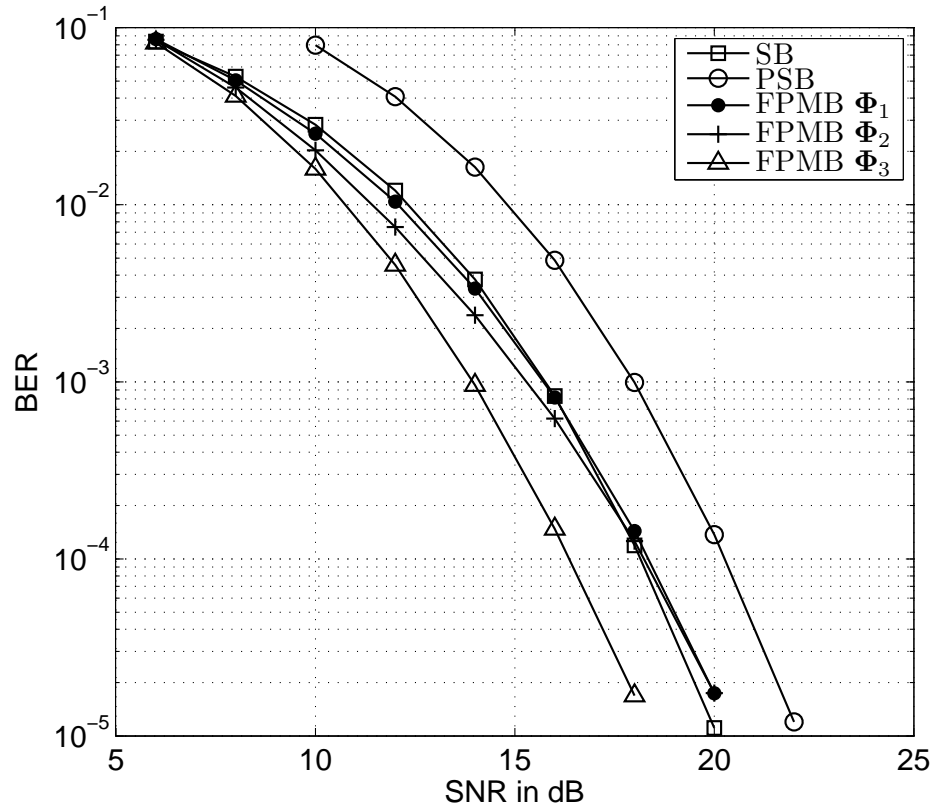


Fig. 3. BER vs. SNR comparison for  $3 \times 3$  64-QAM SB, 4-QAM  $R = 3$  PSB, and 4-QAM  $S = 3$  FPMB.

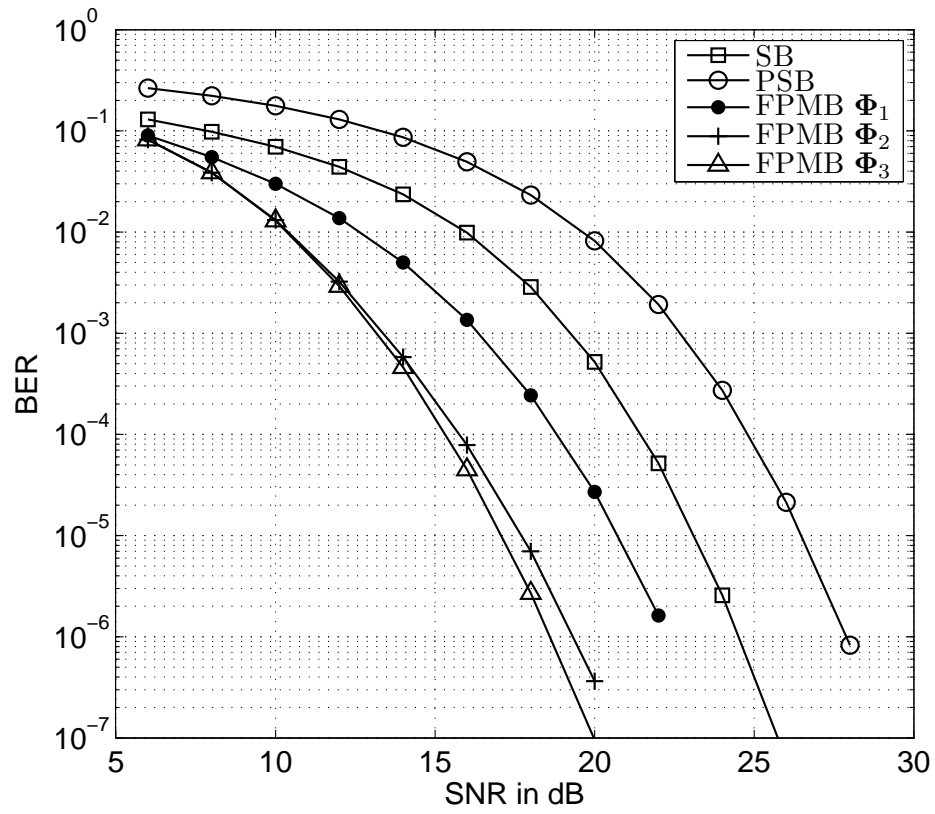


Fig. 4. BER vs. SNR comparison for  $4 \times 4$  256-QAM SB, 4-QAM  $R = 4$  PSB, and 4-QAM  $S = 4$  FPMB.



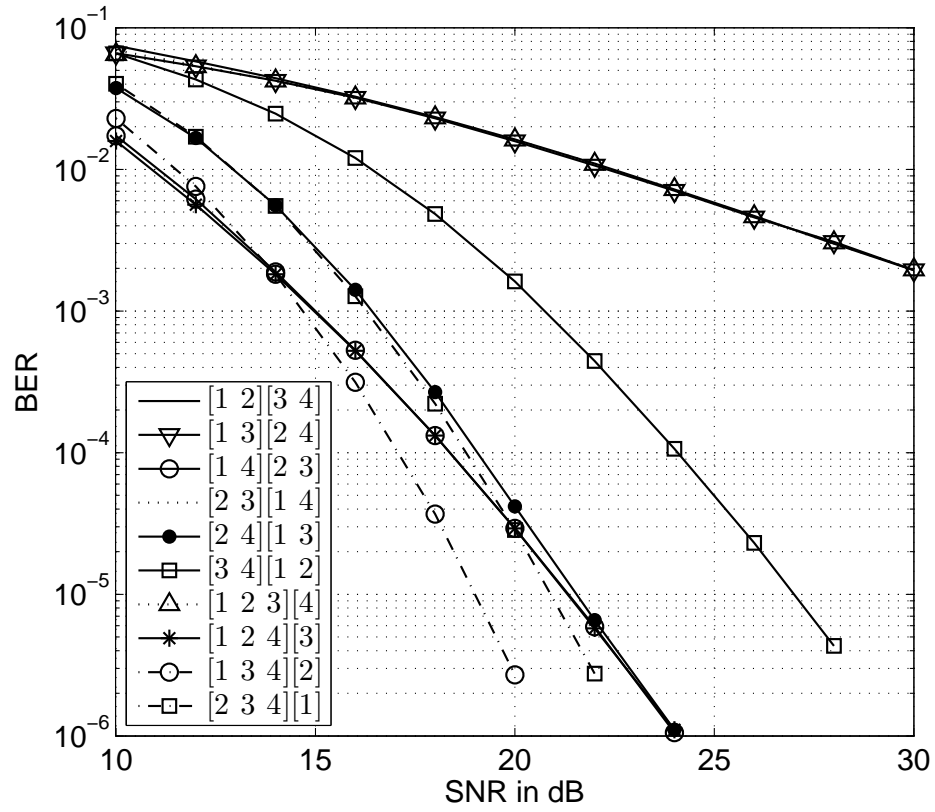


Fig. 5. BER vs. SNR for  $4 \times 4$   $S = 4$ , 4-QAM PPMB.